Exam II
Section I
Part A — No Calculators

1. C p. 25

$f(x) = 4x^3 - 3x - 1 \quad\Rightarrow\quad f(2) = 32 - 6 - 1 = 25 \quad\Rightarrow\quad$ Point $(2,25)$

$f'(x) = 12x^2 - 3 \quad\Rightarrow\quad f'(2) = 48 - 3 = 45 \quad\Rightarrow\quad$ Slope $= 45$

$y - 25 = 45(x-2) \quad\Rightarrow\quad y = 45x - 65$

2. A p. 25

$$\int_0^1 \sin(\pi x)\,dx = -\tfrac{1}{\pi}\cos(\pi x)\Big]_0^1 = -\tfrac{1}{\pi}(\cos\pi - \cos 0) = -\tfrac{1}{\pi}(-1-1) = \tfrac{2}{\pi}$$

3. B p. 26

This is the definition of the derivative of the cosine function at coordinate x. Hence

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \cos'(x) = -\sin x$$

4. C p. 26

The graph is concave up when $y'' > 0$.

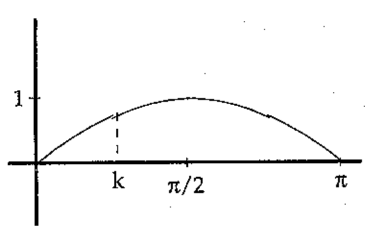
$y = x^5 - 5x^4 + 10x + 15$

$y' = 5x^4 - 20x^3 + 10$

$y'' = 20x^3 - 60x^2 = 20x^2(x-3)$

The factor $(20x^2)$ is always positive. The sign of y'' depends upon the sign of the factor $(x-3)$. $y'' > 0$ if and only if $x > 3$.

5. D p. 26



$$\int_0^\pi \sin x\,dx = -\cos x\Big]_0^\pi = -\cos\pi + \cos 0 = 2.$$

Thus the whole region has area 2.

We need to divide this into two parts, with the left-hand section having an area that is $\frac{1}{4}$ of the whole region. Hence we must find k so that $\int_0^k \sin x\,dx = \frac{1}{4}(2) = \frac{1}{2}$.

This gives: $-\cos x\Big]_0^k = \frac{1}{2} \quad\Rightarrow\quad -\cos k + 1 = \frac{1}{2} \quad\Rightarrow\quad \cos k = \frac{1}{2} \quad\Rightarrow\quad k = \frac{\pi}{3}$

6. C p. 27

$$\begin{aligned} x(t) &= (t-2)^3\,(t-6) \\ x'(t) &= 3(t-2)^2\,(t-6) + (t-2)^3 \\ &= (t-2)^2\,[\,3(t-6) + (t-2)\,] \\ &= (t-2)^2\,(4t-20) \\ &= 4(t-2)^2\,(t-5) \end{aligned}$$

This is positive-valued when $t > 5$.

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7. E p. 27

The antiderivative of sec must be memorized.

$$\frac{dy}{dx} = \sec x \quad \Rightarrow \quad y = \ln\left|\sec x + \tan x\right| + C$$

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8. A p. 27

Note that the numerator is the derivative of the denominator. Hence

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx = \ln\left|\tan x\right|\Big]_{\pi/4}^{\pi/3} = \ln\left|\tan\frac{\pi}{3}\right| - \ln\left|\tan\frac{\pi}{4}\right|$$

$$= \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}.$$

This integration problem can also be done with a formal substitution.

Let $u = \tan x$. Then $du = \sec^2 x\,dx$.

In addition, since this is a definite integral, we can change the limits of integration.
When $x = \frac{\pi}{4}$, then $u = \tan\frac{\pi}{4} = 1$.
When $x = \frac{\pi}{3}$, then $u = \tan\frac{\pi}{3} = \sqrt{3}$.

Hence $$\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx = \int_1^{\sqrt{3}} \frac{du}{u} = \ln|u|\Big]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}.$$

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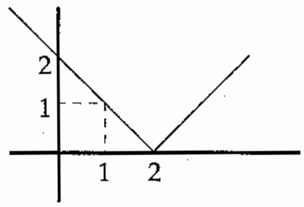
9. B p. 28

$$\lim_{x\to\infty} \frac{x^2-6}{2+x-3x^2} = \lim_{x\to\infty} \frac{1-\frac{6}{x^2}}{\frac{2}{x^2}+\frac{1}{x}-3} = -\frac{1}{3}.$$

D p. 28

$$\int_0^2 \sqrt{x^2 - 4x + 4}\,dx = \int_0^2 \sqrt{(x-2)^2}\,dx = \int_0^2 |x-2|\,dx$$

To evaluate this integral, count squares in the graph at the right, or note that the area of the triangle is $\frac{1}{2}(2)(2) = 2$.



C p. 28

$g(x) = \dfrac{x-2}{x+2}$

$g'(x) = \dfrac{(x+2)\cdot 1 - (x-2)\cdot 1}{(x+2)^2} = \dfrac{4}{(x+2)^2}$

Hence $g'(2) = \dfrac{4}{4^2} = \dfrac{1}{4}$.

B p. 29

$\dfrac{dy}{dx} = 2xy$

$\dfrac{1}{y}\,dy = 2x\,dx$

$\ln|y| = x^2 + C$

$|y| = e^{x^2 + C} = e^{x^2}e^C \quad \Rightarrow \quad y = De^{x^2}$

Then $x = 0,\ y = 4 \quad \Rightarrow \quad 4 = De^0 \quad \Rightarrow \quad D = 4 \quad \Rightarrow \quad y = 4e^{x^2}$

A p. 29

$f(x) = (2x-3)^4$

$f'(x) = 4(2x-3)^3 \cdot 2$

$f''(x) = 4 \cdot 3(2x-3)^2 \cdot 2^2$

$f'''(x) = 4 \cdot 3 \cdot 2(2x-3) \cdot 2^3$

$f^{(4)}(x) = 4! \cdot 2^4$

14. D p. 29

$$\int_2^4 (f(x)+3)\,dx = \int_2^4 f(x)\,dx + \int_2^4 3\,dx$$
$$= 6 + 3\cdot 2 = 12$$

15. E p. 30

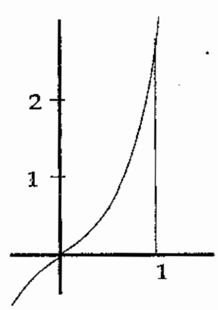
$$\tan(x+y) = x$$
$$\sec^2(x+y)\left[1+\frac{dy}{dx}\right] = 1$$
$$1+\frac{dy}{dx} = \cos^2(x+y)$$
$$\frac{dy}{dx} = \cos^2(x+y) - 1$$

16. C p. 30

Solution I. $f(g(x)) = f(\ln x) = e^{2\ln x} = e^{\ln x^2} = x^2$
Then the derivative of $f(g(x))$ is $2x$.
The value of the derivative at $x = e$ is $2e$.

Solution II. $D_x[f(g(x))] = f'(g(x))\cdot g'(x)$.
At $x = e$, $g(x) = 1$.
Hence we want $f'(1)\cdot g'(e)$.
$f'(x) = 2e^{2x}$ and $g'(x) = \frac{1}{x}$, so $f'(1) = 2e^2$ and $g'(e) = \frac{1}{e}$.
Thus $D_x[f(g(x))]$, at $x = e$, has the value $2e^2\cdot\frac{1}{e} = 2e$.

17. C p. 30



$$\int_0^1 x\,e^{x^2}\,dx = \frac{1}{2}\int_0^1 (2x)\,e^{x^2}\,dx$$
$$= \frac{1}{2}\,e^{x^2}\Big]_0^1$$
$$= \frac{1}{2}(e-1)$$

18. E p. 31

$$h'(x) = \frac{3}{4}(x^2-4)^{-1/4}\cdot(2x) = \frac{6}{4(x^2-4)^{1/4}}$$
Then calculating $h'(2)$ involves a 0 in the denominator and a nonzero numerator, hence, $h'(2)$ does not exist.

19. B p. 31

$$y = \sqrt{x} - \frac{1}{x\sqrt[3]{x}} = x^{1/2} - x^{-4/3} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{4}{3}x^{-7/3}$$

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20. D p. 31

In order for f to be continuous, we must have $\lim\limits_{x\to1} f(x) = f(1)$.

$$\begin{aligned}
\lim_{x\to1} f(x) &= \lim_{x\to1} \frac{\sqrt{x+3} - \sqrt{3x+1}}{x-1} \\
&= \lim_{x\to1} \frac{\sqrt{x+3} - \sqrt{3x+1}}{x-1} \cdot \frac{\sqrt{x+3} + \sqrt{3x+1}}{\sqrt{x+3} + \sqrt{3x+1}} \\
&= \lim_{x\to1} \frac{(x+3)-(3x+1)}{(x-1)(\sqrt{x+3} + \sqrt{3x+1})} = \lim_{x\to1} \frac{-2x+2}{(x-1)(\sqrt{x+3} + \sqrt{3x+1})} \\
&= \lim_{x\to1} \frac{-2(x-1)}{(x-1)(\sqrt{x+3} + \sqrt{3x+1})} = \lim_{x\to1} \frac{-2}{\sqrt{x+3} + \sqrt{3x+1}} = -\frac{1}{2}
\end{aligned}$$

Thus k must equal $-\frac{1}{2}$.

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21. D p. 32

$$f(x) = \frac{x}{2x-3}$$

$$f'(x) = \frac{(2x-3) - x\cdot 2}{(2x-3)^2}$$

$$f'(1) = \frac{-1-2}{(-1)^2} = -3$$

The normal is a line that is perpendicular to the tangent at a point. Since the tangent line at $x = 1$ has a slope of -3, then the slope of the normal there must be $\frac{1}{3}$. Since $f(1) = -1$, the point at which the normal is to be drawn is $(1,-1)$.

Thus the equation of the line is:

$$\begin{aligned}
y+1 &= \tfrac{1}{3}(x-1) \\
3y+3 &= x-1 \\
4 &= x-3y
\end{aligned}$$

22. A p. 32

With $f(x) = x \ln x$, we have $f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$.
The critical number occurs when $f'(x) = 0$.
$1 + \ln x = 0 \quad \Rightarrow \quad \ln x = -1$
$\Rightarrow \quad x = e^{-1}$
Evaluating the function f at $x = e^{-1}$, we have:
$f(e^{-1}) = e^{-1} \ln(e^{-1}) = e^{-1} \cdot (-1) = -\frac{1}{e}$.
To be sure that this is the <u>minimum</u> value, we check the concavity of the curve at $x = e^{-1}$.
$f''(x) = \frac{1}{x}$, so $f''(x)$ has a positive value at the positive number $x = e^{-1}$.
Hence the curve is concave up at the critical number $x = e^{-1}$, so there is a minimum value achieved there.

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23. C p. 33

The slope field shows segments all with the same slope for a given y. That is, the slopes do no vary as x changes. Hence any suggested solution that has x in the formula for $\frac{dy}{dx}$ is incorrect. That allows the elimination of three of the proposed answers. If the correct answer were $\frac{dy}{dx} = y^2$, then the slopes of the segments would be at least 0 everywhere. They are not. Hence the correct answer is $\frac{dy}{dx} = -y$.

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24. B p. 33

$$\frac{1}{\pi/4}\int_0^{\pi/4} \sec^2 x \, dx = \frac{4}{\pi} \tan x \Big]_0^{\pi/4} = \frac{4}{\pi} \cdot 1 = \frac{4}{\pi}$$

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25. B p. 33

$g'(x) > 1$ assures us that g is strictly increasing. Then g will be one-to-one, and therefore will have an inverse.

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26. E p. 34

Since f is continuous, positive at $x = 4$ and negative at $x = 5$, by the Intermediate Value Theorem, there is a point in the interval $[4, 5]$ where the function value is 0.

27. A p. 34

With $G(x) = \int_0^{2x} \cos(t^2)\,dt$, we first let $F(x) = \int_0^{x} \cos(t^2)\,dt$.

Then by the Second Fundamental Theorem, $F'(x) = \cos(x^2)$.

In addition, $G(x) = F(2x)$.

We use the Chain Rule to differentiate this.

$G'(x) = F'(2x) \cdot 2$.

Since $F'(x) = \cos(x^2)$, we have $F'(2x) = \cos((2x)^2) = \cos(4x^2)$.
Thus $G'(x) = 2\cos(4x^2)$ and $G'(\sqrt{\pi}) = 2\cos(4\pi) = 2$.

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28. E p. 34

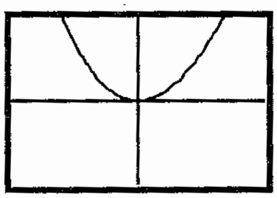
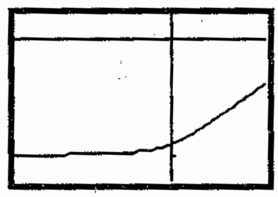
Velocity is the rate of change of the position with respect to time.
Hence we need to determine $\frac{dx}{dt}$.
Differentiate the given expression implicitly with respect to t.
$t \cdot \frac{dx}{dt} + x = 2x\frac{dx}{dt} \quad \Rightarrow \quad x = (2x - t)\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{x}{2x - t}$
When the particle is at $x = 2$, the time is determined by the equation
$tx = x^2 + 8$; in particular, $2t = 12$, so $t = 6$.
Then the velocity is evaluated by substituting $x = 2$ and $t = 6$ into the expression for $\frac{dx}{dt}$.
$\frac{dx}{dt} = \frac{2}{4 - 6} = -1$.

Exam II
Section I
Part B — Calculators Permitted

1. E P. 35

A function has a derivative at a particular x-coordinate if its graph is smooth there.

I. $y = |x^3 - 3x^2|$ II. $y = \sqrt{x^2 + .01} - |x - 1|$ III. $y = \frac{e^x}{\cos x}$


All three of these graphs are smooth at $x = 0$.

2. D p. 35

$$\text{Volume} = \int_0^5 20\,e^{.02t}\,dt = \frac{20}{.02}\,e^{.02t}\Big]_0^5 = 1000\,(e^{.1} - e^0) \approx 105 \text{ gal}$$

3. D p. 36

By the Quotient Rule, $f'(x) = \frac{(a + x^3)\cdot 6 - (6x)\cdot(3x^2)}{(a + x^3)^2} = \frac{6a - 12x^3}{(a + x^3)^2}$.

Then $f'(0) = \frac{6a}{a^2} = \frac{6}{a}$.

We are given that $f'(0) = 3$. Hence $\frac{6}{a} = 3$, so $a = 2$.

4. C p. 36

$$f(x) = \frac{(x-1)^2}{2x^2 - 5x + 3} = \frac{(x-1)^2}{(2x-3)(x-1)} = \frac{x-1}{2x-3} \quad \text{if } x \neq 1.$$

I. $f(1)$ does not exist; hence f is **not** continuous at $x = 1$. False

II. $\lim_{x\to 1} f(x) = 0$; thus there is **not** a vertical asymptote at $x = 1$.
There is a vertical asymptote at $x = \frac{3}{2}$. False

III. $\lim_{x\to\infty} f(x) = \frac{1}{2}$ True

5. C p. 36

Solution I. By the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Since $\frac{dy}{du} = 1 + 2e^u$ and $\frac{du}{dx} = \frac{1}{x}$, then $\frac{dy}{dx} = (1 + 2e^u) \cdot \frac{1}{x}$.

When $x = \frac{1}{e}$, we have $u = 1 + \ln\frac{1}{x} = 1 - 1 = 0$.

With those values: $\frac{dy}{dx} = (1 + 2e^0) \cdot e = 3e$

Solution II. Substitution gives $y = (1 + \ln x) + 2e^{1 + \ln x} = 1 + \ln x + 2ex$
Then $\frac{dy}{dx} = \frac{1}{x} + 2e$, so when $x = \frac{1}{e}$, $\frac{dy}{dx} = 3e$.

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6. B p. 37

Since the radius is always 3 times the height, , we have $h = \frac{1}{3}r$, so
$V = \frac{1}{3}\pi r^2(\frac{1}{3}r) = \frac{1}{9}\pi r^3$. Then $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$.
On the graph, we see that when $t = 6$, then $V = 4\pi$. In addition, we can approximate the slope of the curve, $\frac{dV}{dt}$, at the point where $t = 6$.
$\frac{dV}{dt} \approx \frac{1.6\pi}{2} = 0.8\pi$.
Knowing that the volume is 4π when $t = 6$ allows us to compute the radius of the conical pile at that moment. $4\pi = \frac{1}{9}\pi r^3 \Rightarrow r = 36^{1/3}$.
Substituting these values into the equation $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$, we have:
$0.8\pi = \frac{1}{3}\pi\, 36^{2/3}\frac{dr}{dt}$, which gives $\frac{dr}{dt} = \frac{2.4\pi}{36^{2/3}\pi} \approx 0.22$

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7. E p. 37

I. $f(1) = 1, \quad f'(3) \approx -2$ True

II. $\int_1^2 f(x)\,dx \approx 6.5, \quad f'(3.5) = 0$ True

III. $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = f'(2) \approx 1$
$\frac{f(2.5) - f(2)}{2.5 - 2} \approx \frac{2.5 - 2.3}{2.5 - 2} = \frac{.2}{.5} = .4$ True

8. A p. 37

Solution I. Starting with $5x^3 + 40 = \int_a^x f(t)\,dt$, differentiate to obtain $15x^2 = f(x)$.

Then $\int_a^x 15t^2\,dt = 5t^3\Big]_a^x = 5x^3 - 5a^3$

Since we are given that $5x^3 + 40 = \int_a^x f(t)\,dt$,

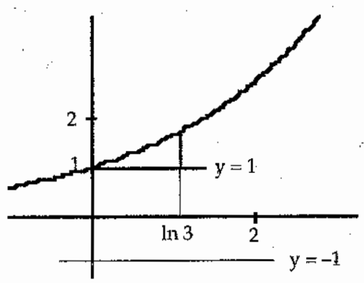
we can set $5x^3 + 40 = 5x^3 - 5a^3$.

Then $-5a^3 = 40$, so $a = -2$.

Solution II. Let $x = a$, then $5a^3 + 40 = \int_a^a f(t)\,dt = 0$.

Then $5a^3 + 40 = 0$, and $a = -2$.

9. A p. 38



Use the washer method.

$$V = \pi\int_0^{\ln 3}\left[\left(e^{x/2}+1\right)^2 - 2^2\right]dx$$

$$\approx 5.128\,.$$

10. A p. 38

$$\frac{dy}{dx} = \frac{x\sin(x^2)}{y}$$

$$y\,dy = x\sin(x^2)\,dx$$

$$\int y\,dy = \int x\sin(x^2)\,dx$$

$$\frac{y^2}{2} = -\frac{1}{2}\cos(x^2) + C$$

$$y^2 = -\cos(x^2) + D$$

$x = 0,\ y = 1 \Rightarrow 1 = -\cos(0) + D \Rightarrow D = 2$

$y = \sqrt{2 - \cos(x^2)}$

11. D p. 38

$y = \sin u \qquad u = v - \frac{1}{v} \qquad v = \ln x$

$\frac{dy}{du} = \cos u \qquad \frac{du}{dv} = 1 + \frac{1}{v^2} \qquad \frac{dv}{dx} = \frac{1}{x}$

$x = e \quad \Rightarrow \quad v = 1 \quad \Rightarrow \quad u = 0$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\ &= (\cos u)\left(1 + \frac{1}{v^2}\right)\left(\frac{1}{x}\right) \\ &= 1 \cdot 2 \cdot \frac{1}{e} = \frac{2}{e} \end{aligned}$$

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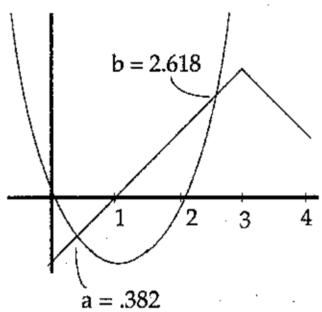
12. A p. 39



The area is the definite integral of the difference of the functions over the interval $[a, b]$. Use the graphing calculator to determine that $a = 0.382$ and $b = 2.618$.

Then letting $y_1 = 2 - |x - 3|$ and $y_2 = x^2 - 2x$ we have

$$\int_a^b (y_1 - y_2)\, dx \approx 1.863 .$$

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13. D p. 39

I. The function f has a horizontal tangent at each x-coordinate where $f'(x) = 0$. This is the case at $x = -3, 1,$ and 3, but **not** at $x = 2$. False

II. Since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, the function f is decreasing to the left of $x = 1$ and increasing to the right. True

III. Since $f'(x)$ is increasing to the left of $x = -3$ and decreasing to the right of $x = -3$, the concavity if the graph of f will change at $x = -3$. True

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14. D p. 40

$$\begin{aligned} f(x) &= (x^2 - 5)^3 \\ f'(x) &= 3(x^2 - 5)^2 (2x) \\ f''(x) &= 6(x^2 - 5)(4x^2) + 3(x^2 - 5)^2 \cdot 2 \\ &= 3(x^2 - 5)[8x^2 + 2(x^2 - 5)] \\ &= 3(x^2 - 5)(10x^2 - 10) \\ &= 30(x + \sqrt{5})(x - \sqrt{5})(x + 1)(x - 1) \end{aligned}$$

The second derivative has sign changes at four different x-coordinates.

15. C p. 40

$N(t) = 200 \ln(t^2 + 36)$

$N'(t) = 200 \cdot \dfrac{1}{t^2 + 36} \cdot 2t$

The graph of $N'(t)$ has a maximum at $t = 6$ days.

Alternatively, $N''(t) = \dfrac{(t^2 + 36)(400) - 400t(2t)}{(t^2 + 36)^2} = \dfrac{14400 - 400t^2}{(t^2 + 36)^2}$.

$N''(t) = 0$ when $t^2 = 36$ and $t = 6$ days.

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16. C p. 41

$a(t) = 4e^{2t} \quad \Rightarrow \quad v(t) = 2e^{2t} + C$

When $t = 0$, then $v = -2$.

In the last equation, this gives $-2 = 2e^0 + C$, so that $C = -4$.

Then $v(t) = 2e^{2t} - 4$.

Antidifferentiating again, we have $x(t) = e^{2t} - 4t + D$.

When $t = 0$, then $x = 2$.

This gives $2 = e^0 - 4 \cdot 0 + D$, so that $D = 1$.

Then $x(t) = e^{2t} - 4t + 1$, and we obtain the particular value $x(\frac{1}{2}) = e - 1$.

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17. D p. 41

I.	On the interval $(-2,-1)$, $f'(x) > 0$. Hence f is increasing.	True
II.	At $x = 0$, $f'(x)$ changes from decreasing to increasing.	True
III.	On the interval $(-1, 0)$, $f'(x)$ is decreasing, so f is concave down.	False

Exam II
Section II
Part A — Calculators Permitted

1. p. 43

The curves intersect at $A = 1.102$.

1: Correct limits in an integral in (a), (b) and (c)

(a) $\displaystyle\int_0^A [(4-x^2)-(1+2\sin x)]\,dx \approx 1.764$

$2:\begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) $\displaystyle\pi\int_0^A \left[(4-x^2)^2 - (1+2\sin x)^2\right] dx \approx 30.460$

$3:\begin{cases} 2: \text{integrand} + \text{constant} \\ 1: \text{answer} \end{cases}$

(c) $\displaystyle\int_0^A \left[(4-x^2)-(1+2\sin x)\right]^2 dx \approx 3.671$

$3:\begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

2. p. 44

(a) Let the side of the square be denoted by x.
Since the perimeter $P = 4x$,
we have $\frac{dP}{dt} = 4\frac{dx}{dt}$.

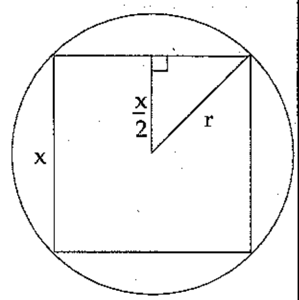
Hence $\quad \frac{dp}{dt} = 8 \Rightarrow \frac{dx}{dt} = 2.$

In addition, in the isosceles right triangle in the figure to the right,

$r = \frac{x}{2}\sqrt{2} = \frac{x}{\sqrt{2}}.$

$r = \frac{x}{\sqrt{2}} \Rightarrow \frac{dr}{dt} = \frac{1}{\sqrt{2}}\frac{dx}{dt} = \sqrt{2} = 1.414$

$C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi\sqrt{2} = 8.886 \text{ in/sec}$



$5:\begin{cases} 2: \text{Equation relating perimeter } P \text{ and side } x \text{ of square} \\ 1: \frac{dP}{dt} \\ 1: \frac{dx}{dt} \\ 1: \text{answer} \end{cases}$

(b) $A = \pi r^2 - (r\sqrt{2})^2$
$\quad = (\pi - 2)\,r^2$

When the square has side 4, the circle has radius $2\sqrt{2}$.

$\frac{dA}{dt} = (\pi-2)\,2r\,\frac{dr}{dt}$
$\quad = (\pi-2)\,(2\cdot 2\sqrt{2})\,\sqrt{2} = 8(\pi-2) = 9.133 \text{ in}^2/\text{sec}$

$4:\begin{cases} 2: \text{equation relating are} \\ 1: \frac{dA}{dt} \\ 1: \text{answer} \end{cases}$

3. p. 45

(a) $F(x) = \int_1^{\sqrt{x}} \frac{2t-1}{t+2}\,dt \quad \Rightarrow \quad F(1) = \int_1^1 \frac{2t-1}{t+2}\,dt = 0.$

1: answer

(b) $F'(x) = \frac{2\sqrt{x}-1}{\sqrt{x}+2} \cdot \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad F'(1) = \frac{2-1}{1+2} \cdot \frac{1}{2} = \frac{1}{6}.$

$2: \begin{cases} 1: F'(x) \\ 1: F'(1) \end{cases}$

(c) slope $= F'(1)$ and $F(1) = 0$

$y - 0 = \frac{1}{6}(x-1)$

$3: \begin{cases} 1: \text{slope} \\ 1: \text{point} \\ 1: \text{tangent equation} \end{cases}$

(d) F increases when $F'(x) > 0$.

$F'(x) = \frac{2\sqrt{x}-1}{\sqrt{x}+2} \cdot \frac{1}{2\sqrt{x}} = 0$ when $2\sqrt{x} - 1 = 0$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

F increases when $x > \frac{1}{4}$.

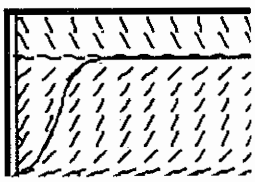
$3: \begin{cases} 1: F'(x) > 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

Exam II
Section II
Part B — No Calculators

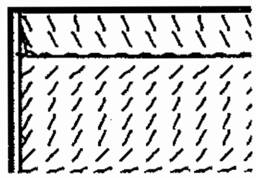
4. p. 46

Logistic differential equation $\frac{dP}{dt} = 2P(3-P) = 6P\left(1-\frac{P}{3}\right)$ with carrying capacity $L = 3$ and growth constant $k = 6$.

(a) Starting at $(0, 0.1)$:



Starting at $(0, 4.0)$:



The solution that starts at $(0, 0.1)$ has an inflection point.

$3: \begin{cases} 1: \text{1st solution curve} \\ 1: \text{2nd solution curve} \\ 1: \text{inflection point} \end{cases}$

(b) The slope of the tangent to the curve $y = P(t)$ at $(0, 1)$ is:

$$m = \frac{dP}{dt} = 2P(3-P)\Big|_{(0,1)} = 2(3-1) = 4.$$

The equation of the tangent line is: $y - 1 = 4(x - 0)$ or $y = 4x + 1$.

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent equation} \end{cases}$

(c) $P(t) = \dfrac{3}{1+2e^{-6t}}$ and $P'(t) = -3\left(1+2e^{-6t}\right)^{-2}\left(-12e^{-6t}\right) = \dfrac{36e^{-6t}}{\left(1+2e^{-6t}\right)^2}$

Substituting in the equation $P'(t) = 2P(3-P)$ we have

$$\frac{36e^{-6t}}{\left(1+2e^{-6t}\right)^2} \quad | \quad 2\cdot\left(\frac{3}{1+2e^{-6t}}\right)\left(3-\frac{3}{1+2e^{-6t}}\right)$$

$$| \quad 2\cdot\left(\frac{9}{1+2e^{-6t}} - \frac{9}{\left(1+2e^{-6t}\right)^2}\right)$$

$$= \frac{36e^{-6t}}{\left(1+2e^{-6t}\right)^2}$$

$2: P(t) = \dfrac{3}{1+2e^{-6t}}$ is a solution

(d) Solution 1: $\dfrac{dP}{dt} = 2P(3-P) = 6P - 2P^2$

$$\frac{d^2P}{dt^2} = (6-4P)\frac{dP}{dt} = (6-4P)(2P)(3-P)$$

$$\frac{d^2P}{dt^2} = 0, \text{ when } P = \frac{3}{2}, 0, 3.$$

Since the graph of $\dfrac{dP}{dt}$ vs P is a concave down parabola with zeros at 0 and 3, the maximum is at $P = \dfrac{3}{2}$.

Solution 2:

Maximum growth rate occurs at $\dfrac{1}{2}$ carrying capacity $\frac{1}{2}(3) = \frac{3}{2}$.

2: answer

5. p. 47

(a) F is increasing on (–3,1) and (3,4) since $F'(x) > 0$ on the intervals.

$3: \begin{cases} 2\text{: answer} \\ 1\text{: justification} \end{cases}$

(b) The absolute minimum occurs at $x = -3$. F increases a great deal from then until $x = 1$. Then it decreases a little (until $x = 3$), and then increases again until $x = 4$.
The absolute maximum occurs either at $x = 1$ or $x = 4$. It depends upon the relative sizes of the regions between the x-axis and the curve on the intervals (1,3) and (3,4).

$3: \begin{cases} 1\text{: answer} \\ 2\text{: reasoning} \end{cases}$

(c) F is concave down if F' is decreasing. This occurs on the interval (–2, 2).

2: answer

6. p. 48

(a) $f(x) = ax + \frac{b}{x}$, where a and b are positive.

$$f'(x) = a - \frac{b}{x^2}$$

$$f'(x) > 0 \quad \Rightarrow \quad ax^2 - b > 0$$

$$\Rightarrow \quad x^2 > \frac{b}{a}$$

$$\Rightarrow \quad x < -\sqrt{\frac{b}{a}} \quad \text{or} \quad x > \sqrt{\frac{b}{a}}$$

$2: \begin{cases} 1\text{: } f'(x) > 0 \\ 1\text{: answer} \end{cases}$

(b) There is a relative maximum at $x = -\sqrt{\frac{b}{a}}$; $y = a\left[-\sqrt{\frac{b}{a}}\right] + \frac{b}{-\sqrt{\frac{b}{a}}}$

$$= -2\sqrt{ab}.$$

There is a relative minimum at $x = \sqrt{\frac{b}{a}}$; $y = a\left[\sqrt{\frac{b}{a}}\right] + \frac{b}{\sqrt{\frac{b}{a}}}$

$$= 2\sqrt{ab}.$$

$3: \begin{cases} 1\text{: critical points} \\ 2\text{: answer} \end{cases}$

(c) $f''(x) = \frac{2b}{x^3} > 0$ when $x > 0$.

$2: \begin{cases} 1\text{: } f''(x) > 0 \\ 1\text{: answer} \end{cases}$

(d) $f''(x)$ never changes sign at a point on the curve. Although the curve is concave down for $x < 0$ and the curve is concave up for $x > 0$, there is not a point of inflection at $x = 0$ since there is no point on the curve there.

$2: \begin{cases} 1\text{: answer} \\ 1\text{: justification} \end{cases}$